Name of college - S.S. college, J Bad

Deptt - Mathematics

Topic - Problem on Rolle's theorem (Real Analysis)

Class - B.Sc part-II

Time - 7.30 A.M to 9.A.M

Date - 26-04-2021

Teacher's Name-

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Problem. Problem on Rolle's and Mean value theorem

Verify Rolle's theorem over $[a,b]$ for the function

$$f(x) = (x-a)^m (x-b)^n$$

Solution :→ By the question

$$f(x) = (x-a)^m (x-b)^n$$

$$\Rightarrow f(a) = 0 \qquad \text{thus } f(a) = f(b) = 0$$

$$f(b) = 0$$

By ordinary Differentiation

$$f'(x) = m(x-a)^{m-1}(x-b)^n + n(x-a)^m (x-b)^{n-1} \quad \text{—— (1)}$$

$$Rf'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h-a)^m (x+h-b)^n - (x+a)^m (x-b)^n}{h}$$

$$= \lim_{h\to 0} \frac{\{(x-a)+h\}^m \{(x-b)+h\}^n - (x+a)^m (x-b)^n}{h}$$

$$= \lim_{h\to 0} \frac{\left\{(x-a)^m + m C_1 (x-a)^{m-1} h + \cdots\right\}\left\{(x-b)^n + n C_1 (x-b)^{n-1} h \cdots\right\} - (x-a)^m (x-b)^n}{h}$$

$$= n(x-a)^m (x-b)^{n-1} + m(x-a)^{m-1}(x-b)^n$$

$$Lf'(x) = \lim_{h\to 0} \frac{f(x-h) - f(x)}{-h}$$

$$= \lim_{h\to 0} \frac{(x-h-a)^m (x-h-b)^n - (x-a)^m (x-b)^n}{-h}$$

$$= \lim_{h\to 0} \frac{\{(x-a)-h\}^m \{(x-b)-h\}^n - (x-a)^m (x-b)^n}{-h}$$

$$= \lim_{h\to 0} \frac{\left\{(x-a)^m - m C_1 (x-a)^{m-1} h + \cdots\right\}\left\{(x-b)^n - n C_1 (x-b)^{n-1} h + \cdots\right\} - (x-a)^m (x-b)^n}{-h}$$

$$= \left\{-n(x-a)^m (x-b)^{n-1} - m(x-a)^{m-1}(x-b)^n\right\}(-1)$$

$$= n(x-a)^m (x-b)^{n-1} + m(x-a)^{m-1}(x-b)^n$$

thus $Rf'(x) = Lf'(x)$

$\Rightarrow f'(x)$ exists for all values of x in $[a,b]$

Since $f(x)$ is differentiable in $[a,b]$

$\Rightarrow f(x)$ is also continuous in each point of $[a,b]$

Hence All the three Required Conditions of Rolle's theorem are satisfied

$\therefore f'(x)=0$ for at least one value of x in $[a,b]$

$\Rightarrow m(x-a)^{m-1}(x-b)^n + n(x-a)^m(x-b)^{n-1} = 0$

$\Rightarrow (x-a)^{m-1}(x-b)^{n-1}\{m(x-b)+n(x-a)\} = 0$

$\Rightarrow x=a$
$x=b$ $\quad x = \dfrac{na+mb}{m+n}$

~~evidently~~ evidently
$x = \frac{na+mb}{m+n}$ means the point
Which divides $[a,b]$ in the ratio $m:n$
and as lies in $[a,b]$ and.

2. Discuss applicability of Rolle's theorem to the function $f(x)=|x|$ in $[-1,1]$

By the question, $f(x)=|x|$

$\Rightarrow f(1) = |1| = 1$

$f(-1) = |-1| = 1$

$f(x)=|x|$ is continuous for all values of x in $(-1,1)$

Also $Rf'(0) = \lim_{h\to 0} \dfrac{f(0+h)-f(0)}{h}$

$= \lim_{h\to 0} \dfrac{|0+h|-|0|}{h}$

$= 1$

$Lf'(0) = \lim_{h\to 0} \dfrac{f(0-h)-f(0)}{-h}$

$= \lim_{h\to 0} \dfrac{h-0}{-h} = -1$

Here $Rf'(0) \neq Lf'(0)$
So $f'(0)$ does not Exist.
Hence the function is not differentiable in the entire interval $(0,2)$
and so Rolle's theorem is not applicable to the Given function $f(x)$ in $[-1,1]$

Problem :→ Discuss the applicability of Rolle's theorem to the function

$$f(x) = x^2+1 \quad \text{when } 0 \le x \le 1$$
$$= 3-x \quad \text{when } 1 < x \le 2$$

Solution :→ By the question, function $f(x)$ is defined as

$$f(x) = x^2+1 \quad \text{when } 0 \le x \le 1$$
$$= 3-x \quad \text{when } 1 < x \le 2$$

Here $f(0) = 0^2+1 = 1$

$f(2) = 3-2 = 1$

$\therefore f(0) = f(2) = 1$

Here we observe that the $f(x)$ is continuous for all x in the $(0,2)$ Except perhaps at $x=1$

Test the continuity of $f(x)$ at $x=1$

$f(1+h) = 3-(1+h) \quad \therefore \lim_{h\to 0} f(1+h) = \lim_{h\to 0} 2-h = 2$

$f(1-h) = (1-h)^2+1 \quad \therefore \lim_{h\to 0} f(1-h) = \lim_{h\to 0} (1-h)^2+1$

$f(1) = 1^2+1 = 2. \qquad = 1+1 = 2$

Here we see $\lim_{h\to 0} f(1+h) = \lim_{h\to 0} f(1-h) = f(1) = 2$

$\Rightarrow f(x)$ is continuous at $x=1$

and this continuous in the whole interval $(0,2)$

Again $f'(x) = 2x \quad \text{when } 0 \le x \le 1$

$= -1 \quad \text{when } 1 < x \le 2$

$\therefore f(x)$ is differentiable in interval $(0,2)$

Except perhaps at $x=1$

Now $Rf'(1) = \lim_{h\to 0} \dfrac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \dfrac{\{3-(1+h)\}-2}{h}$

$= \lim_{h\to 0} \dfrac{2-h-2}{h} = -1$

Again $Lf'(1) = \lim_{h\to 0} \dfrac{f(1-h) - f(1)}{-h} = \lim_{h\to 0} \dfrac{\{(1-h)^2+1\}-2}{-h}$

$= \lim_{h\to 0} \dfrac{2h-h^2}{-h} = \lim_{h\to 0} 2-h = 2$

$\therefore$ we find $Rf'(1) \neq Lf'(1)$ and so $f'(1)$ does not exist. Hence the function $f(x)$ is not differentiable in the entire interval $(0,2)$ and therefore Rolle's theorem is not applicable to the given function $f(x)$ in $(0,2)$

Construct an example in which the hypothesis as well as conclusion of Rolle's theorem are satisfied

Solution :→ the function $f(x) = \sqrt{a^2 - x^2}$ $a > 0$ is continuous on $[-a, a]$, but is differentiable only on $]-a, a[$

Since its value at $-a$ and a are equal, the conclusion of Rolle's theorem must hold.

In fact $x \neq a, -a$, we find

$$f'(x) = 0$$

$$-\frac{2x}{\sqrt{a^2 - x^2}} = 0$$

$$\Rightarrow x = 0$$

thus there exist a point $c = 0$ in $]-a, a[$ where $f'(c) = 0$